State Space Formulas for a Solution of the Suboptimal Nehari Problem on the Unit Disc

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Abstract

We give state space formulas for a ("central") solution of the suboptimal Nehari problem for functions defined on the unit disc and taking values in the space of bounded operators in separable Hilbert spaces. Instead of assuming exponential stability, we assume a weaker stability concept (the combination of input-, output- and input-output stability), which allows us to solve the problem for general H-infinity functions.

1 Introduction

The Nehari problem on the unit disc can be formulated as follows. Given (separable) Hilbert spaces U and Y and the coefficients $G_n \in \mathcal{L}(U,Y)$ of the power series

$$\sum_{n=1}^{\infty} G_n z^n, \tag{1}$$

find coefficients G_n with $n \leq 0$ such that the Laurent series

$$\sum_{n=-\infty}^{\infty} G_n z^n,$$

defines a $\mathbf{L}_\infty(\mathbb{T};\mathcal{L}(U,Y))$ function on the unit circle. It is well-known that the Hankel operator $H : l^2(\mathbb{N};U) \to l^2(\mathbb{N};Y)$ formed from the problem data as

$$\begin{bmatrix} G_1 & G_2 & G_3 & \dots \\ G_2 & G_3 & & \\ G_3 & & & \\ \vdots & & & \end{bmatrix}$$

plays a crucial role: the problem is solvable if and only if H is a bounded operator and the norm of H equals the infimum of the $\mathbf{L}_\infty$ norms of all the Laurent series solutions. In the scalar case this is due to Nehari [13] and in the

Hilbert space case it is due to Page [17] (see also Nikolskii [14] and Peller [18] for treatments in book form and further references).

The suboptimal Nehari problem is: for a given $\sigma > \|H\|$, parameterize all solutions with $\mathbf{L}_\infty$ norm smaller than or equal to σ. This problem was solved in the scalar case by Adamjan, Arov and Krein [1, 2] and in the Hilbert space case by Kheifets [12] (see also Peller [18]).

The suboptimal Nehari problem has many applications in control theory, e.g. the problem of designing robustly stabilizing controllers (see e.g. Curtain and Zwart [8]). For such applications the above mentioned abstract existence and parametrization results are not enough. More specific information about (at least) one of the solutions is needed in the form of so-called state space formulas, which we explain below.

In the applications to control theory that we have in mind the power series (1) defines a function in the Hardy space $\mathbf{H}_\infty(\mathbb{D};\mathcal{L}(U,Y))$ of the unit disc. This is often called input-output stability. Moreover, there is a realization in terms of operators $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(U,X)$ and $C \in \mathcal{L}(X,Y)$, where the Hilbert space X is called the state space, and $G_n = CA^{n+1}B$, $n > 0$. Whenever the Hankel operator is bounded, there is always a 'trivial' state space realization of (1) called the shift realization with $X = l^2(\mathbb{N};Y)$ and

$$(Ax)_n = x_{n+1}, \quad Bu = (G_n u)_{n\in\mathbb{N}}, \quad Cx = x_1.$$

In fact, there are infinitely many state space realizations. However, in control applications the state space parameters A, B and C are given and they have physical significance. The to be determined operators G_n with $n \leq 0$ should also be given in state space form, since these state space parameters $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ are needed for implementation of the controller. Consequently, in this paper we seek expressions for the to be determined operators $\tilde{A} \in \mathcal{L}(X)$, $\tilde{B} \in \mathcal{L}(U,X)$ and $\tilde{C} \in \mathcal{L}(X,Y)$ in terms of A, B, C such that $G_n = \tilde{C}\tilde{A}^{-n-1}\tilde{B}$ for $n < 0$. This is what we mean by a state space solution to the suboptimal Nehari problem.

Since the state space solutions given in Section 7 involve the expressions

$$L_B = \sum_{k=0}^{\infty} A^{*k}B^*BA^k, \qquad L_C = \sum_{k=0}^{\infty} A^k CC^* A^{*k}, \tag{2}$$

we require that both of these expressions be in $\mathcal{L}(X)$. This is satisfied if and only if the realization is input and output stable (see Section 2). A stronger sufficient condition is that A is exponentially stable, i.e., its spectral radius is strictly less than one. Our three basic assumptions of input stability, output stability and input-output stability are much weaker than the exponential stability assumption.

It is interesting to note that our state space formulas applied to the specific case of the shift realization give an explicit formula for G_n with $n \leq 0$ in terms of the Hankel operator H, the shift τ and the first element e_1 of the standard basis of $l^2(\mathbb{N})$:

$$G_n u = \left(-HH^* \left[\tau^* T_\tau X^{-1}\right]^{-n+1} XHe_1 u\right)_1,$$

with

$$X = (\sigma^2 - HH^*)^{-1},$$

the inverse of the square of the defect operator of H and

$$T_\tau = (\sigma^2 - \tau HH^*\tau^*)^{-1},$$

the inverse of the square of the defect operator of $H^*\tau^*$. Since the Hankel operator is bounded, the condition $L_B, L_C \in \mathcal{L}(X)$ is automatically satisfied for the shift realization. So in the light of the known results on the Nehari problem the condition $L_B, L_C \in \mathcal{L}(X)$ is a natural one. Note also that the above formulas in terms of the Hankel operator make perfect sense even if the power series is not in the Hardy space $\mathbf{H}_\infty$. Although our proof breaks down in this case, we conjecture that the formulas remain valid.

We now compare our results with existing results in the literature on state space formulas for solutions of the suboptimal Nehari problem. There are many results in the half-plane case under the assumption of exponential stability: e.g. Curtain and Zwart [8], Glover et al. [10], Ran [19], Curtain and Ran [7], Curtain and Zwart [3], Curtain and Ichikawa [4]. We generalized these results for systems with realizations that are input stable, output stable and input-output stable in [5], [6, Section 6]. As already mentioned, not every $\mathbf{H}_\infty$ function has an exponentially stable realization, but it does always possess an input stable, output stable and input-output stable realization. The best existing result for the disc case is by Foias et al in [9, Section VI.8] where they assume exponential stability of the state space realization. Their approach uses commutant lifting results which is very different from our approach and unfortunately the formulas are given in a different form. (Their central solution does agree with ours). In contrast, we use the J-spectral factorization approach that we used in the half-plane case [5], [6, Section 6] adapted to the case of the disc. In obtaining the formulas for the J-spectral factorization we were aided by the exposition for the case of rational functions in Ionescu et al [11, Section 8] (in spite of the typo in (8.94) there: $\tilde{R}$ should read $\hat{R}$).

Our paper is arranged as follows. We begin in Section 2 with some background on infinite-dimensional discrete-time state space systems and in Section 3 we introduce the special Riccati equations that lie behind the formulas for the solution to the suboptimal Nehari problem. In Section 4 we introduce and solve the associated J-spectral factorization problem. For notational simplicity in the proofs we assume that $\sigma = 1$. The inverse of the solution to the J-spectral factorization problem found in Section 4 is analyzed in Section 5. These results are then used in Section 6 to obtain an explicit state space solution to the suboptimal Nehari problem for the case $\sigma = 1$. Finally, in Section 7 the solution to the general case (where σ may not equal one) is derived.

In the sequel we shall use the following notation

$$\mathbf{G}(z) = \sum_{n=1}^{\infty} G_n z^n, \qquad \mathbf{Z}(z) = \sum_{n=-\infty}^{0} G_n \frac{1}{z^n}, \qquad \mathbf{K}(z) = \sum_{n=-\infty}^{0} G_n z^n.$$

2 Transfer functions, characteristic functions and stability

We recall some basic facts on infinite-dimensional discrete-time state space systems (a more comprehensive treatment can be found in Opmeer [16]).

For the state space system $\Sigma(A, B, C, D)$ defined by a bounded operator

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in \mathcal{L}\left(\left[\begin{array}{c} X \\ U \end{array}\right], \left[\begin{array}{c} X \\ Y \end{array}\right]\right)$$

with U, X, Y separable Hilbert spaces, we define the transfer function

$$\mathbf{G}(z) = D + \sum_{i=0}^{\infty} CA^i B z^{i+1}.$$

The system is called *input-output stable* if $\mathbf{G}$ is analytic and uniformly bounded on the unit disc, i.e. $\mathbf{G} \in \mathbf{H}_\infty(\mathbb{D}, \mathcal{L}(U, Y))$. The Hankel operator of the system has the infinite matrix representation

$$\begin{bmatrix} CB & CAB & CA^2B & \dots \\ CAB & CA^2B & CA^3B & \dots \\ CA^2B & CA^3B & CA^4B & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Note that all of this is consistent with what was mentioned in the introduction (where $D = 0$).

The *characteristic function* of the state space system is $\mathfrak{G}(z) := Cz(I - zA)^{-1}B + D$, which for $z \neq 0$ may also be written as $C\left(\frac{1}{z} - A\right)^{-1} B + D$. The characteristic function and the transfer function of a state space system are equal for $|z| < 1/r(A)$, where $r(A)$ is the spectral radius of the operator A, but they might have different values at other points (see [8, Example 4.3.8] or [21]). If one assumes exponential stability (also called power stability), i.e. $r(A) < 1$, then $\mathbf{G} = \mathfrak{G}$ on the closed unit disc and the difference is insignificant for the Nehari problem. In this article we do not assume exponential stability and therefore we do have to be careful about this difference. The main advantage of the characteristic function is that it lends itself better for algebraic computations, but it is the transfer function of the system that we are in fact interested in.

At several points in this article we will need to find a state space realization of the inverse of a characteristic function given a realization of that function itself. The following simple lemma will be used for that purpose.

Lemma 2.1. *The characteristic function of the state space system* $\Sigma(A, B, C, D)$ *is invertible if and only if* D *is invertible. In this case it is the characteristic function of the state space system* $\Sigma(A - BD^{-1}C, BD^{-1}, -D^{-1}C, D^{-1})$.

Proof. By multiplying out we obtain

$$\begin{aligned}
&\left[C\left(\frac{1}{z}I-A\right)^{-1}B+D\right]\\
&\quad\times\left[-D^{-1}C\left(\frac{1}{z}I-A+BD^{-1}C\right)^{-1}BD^{-1}+D^{-1}\right]\\
&=I+C\left(\frac{1}{z}I-A\right)^{-1}\\
&\quad\times\left[\left(\frac{1}{z}I-A+BD^{-1}C\right)-\left(\frac{1}{z}I-A\right)-BD^{-1}C\right]\\
&\quad\times\left(\frac{1}{z}I-A+BD^{-1}C\right)^{-1}BD^{-1}\\
&=I.
\end{aligned}$$

That the product in the reverse order is also the identity follows similarly. □

The following stability concept plays an important role in this article. Here $\mathbf{H}_2(\mathbb{D}, Y)$ is the usual Hardy space.

Definition 2.2. *A state space system is called* output stable *if, with*

$$\mathbf{C}(z) := \sum_{i=0}^{\infty} CA^i z^i,$$

for every $x \in X$ we have $\mathbf{C}(\cdot)x \in \mathbf{H}_2(\mathbb{D}, Y)$.

Note that the transfer function of an output stable system, being equal to $D + z\mathbf{C}(z)B$, is analytic in the open unit disc.

Output stability is equivalent to the *observation Lyapunov equation*

$$A^*LA - L + C^*C = 0 \tag{3}$$

having a nonnegative, selfadjoint solution. The *observability gramian* L_C is the smallest nonnegative, selfadjoint solution of the observation Lyapunov equation. An explicit formula for the observability gramian is given in (2).

A state space system is input stable if the dual system $\Sigma(A^*, C^*, B^*, D^*)$ is output stable. The *controllability gramian* L_B of the system is the observability gramian of this dual system. It is is the smallest nonnegative, selfadjoint solution of the control Lyapunov equation

$$ALA^* - L + BB^* = 0.$$

The transfer function of an input stable system is analytic in the open unit disc, just as the transfer function of an output stable system is, as we saw above.

The following key result was first established, in a different context, in Weiss and Weiss [20] for the continuous-time case. See also Oostveen [15, Lemma 4.2.6]. The dashes indicate unimportant entries.

Lemma 2.3. *Let L_C be the observability gramian of the output stable and input-output stable system $\Sigma(A, B, C, -)$. Then $\Sigma(A, -, B^* L_C A, -)$ is output stable.*

Proof. Since the system is input-output stable we have $\mathbf{G} \in \mathbf{H}_\infty(\mathbb{D}, \mathcal{L}(U, Y))$ and since it is output stable we have $\mathbf{C}x \in \mathbf{H}_2(\mathbb{D}, Y)$ for all $x \in X$. It follows that $\mathbf{G}^*\mathbf{C}x \in L^2(\mathbb{T}, U)$ for all $x \in X$ with $\mathbb{T}$ the unit circle. From this it follows that the analytic part of $\mathbf{G}^*\mathbf{C}x$ is in $\mathbf{H}^2(\mathbb{D}, U)$. We calculate for $z \in \mathbb{T}$ (so that $\bar{z} = 1/z$):

$$\mathbf{G}(z)^*\mathbf{C}(z) = \left(\sum_{k=0}^{\infty} B^* A^{*k} C^* z^{-k-1} \right) \left(\sum_{m=0}^{\infty} C A^m z^m \right).$$

It follows that the analytic part is:

$$\sum_{k=0}^{\infty} \sum_{m=k+1}^{\infty} B^* A^{*k} C^* C A^m z^{m-k-1}.$$

Substituting $j = m - k - 1$ we obtain for this analytic part

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} B^* A^{*k} C^* C A^k A^{j+1} z^j,$$

and using that

$$L_C = \sum_{k=0}^{\infty} A^{*k} C^* C A^k,$$

we can rewrite this as

$$\sum_{j=0}^{\infty} B^* L_C A A^j z^j.$$

It follows that the analytic part of $\mathbf{G}^*\mathbf{C}$ equals the '$\mathbf{C}$' function of the system $\Sigma(A, -, B^* L_C A, -)$ defined in Definition 2.2. It follows from this definition and the above that this system is output stable. □

Our stability assumption on the state space realization $\Sigma(A, B, C, D)$ of $\mathbf{G}$ is that it is input stable, output stable and input-output stable.

This assumption is implied by and is strictly weaker than exponential stability. Any function in $\mathbf{H}_\infty(\mathbb{D}, \mathcal{L}(U, Y))$ has a realization that satisfies our stability assumption (for example the shift realization mentioned in the introduction).

3 Algebraic Riccati equations

In this section we introduce two algebraic Riccati equations that play a key role in the solution of the suboptimal Nehari problem.

We first recall some general facts about algebraic Riccati equations.

Definition 3.1. *The control algebraic Riccati equation of a system* $\Sigma(A, B, C, D)$ *is the equation*

$$A^*QA - Q + C^*C = F^*SF,$$

where

$$S := I + D^*D + B^*QB, \quad F := -S^{-1}(B^*QA + D^*C).$$

The closed-loop system corresponding to a nonnegative selfadjoint solution Q *is*

$$\left[\begin{array}{c|c} A+BF & BS^{-1/2} \\ \hline F & S^{-1/2} \\ C+DF & DS^{-1/2} \end{array}\right]. \tag{4}$$

The filter algebraic Riccati equation of a system $\Sigma(A, B, C, D)$ *is the equation*

$$APA^* - P + BB^* = LRL^*,$$

where

$$R := I + DD^* + CPC^*, \quad L := -(APC^* + BD^*)R^{-1}.$$

The following theorem can be proven as was done for the corresponding continuous-time result in [6] (see [16, Proposition 6.34, Corollary 6.40]).

Lemma 3.2. *If the control Riccati equation of* $\Sigma(A, B, C, D)$ *has a nonnegative, selfadjoint solution* Q*, then the closed-loop system* (4) *is output stable and input-output stable. Moreover, the* H_∞ *norm of its transfer function is bounded from above by one. If the filter algebraic Riccati equation of* $\Sigma(A, B, C, D)$ *also has a nonnegative, selfadjoint solution, then the afore-mentioned closed-loop system is also input stable.*

Standing hypothesis From here on, until section 7, we assume that $\mathbf{G} \in \mathbf{H}_\infty(\mathbb{D}, \mathcal{L}(U, Y))$ is such that its Hankel operator has norm strictly smaller than one and $\mathbf{G}(0) = 0$. We take $\sigma = 1$ and fix an input stable, output stable and input-output stable realization $\Sigma(A, B, C, 0)$ of $\mathbf{G}$.

The spectral radius of the product $L_B L_C$ of the controllabilty and observability gramians equals the square of the norm of the Hankel operator $H_\mathbf{G}$ (see [16, Lemma 3.1.8]).

Since we take $\sigma = 1 > \|H_\mathbf{G}\|$ it follows that $I - L_B L_C$ has a bounded inverse and that, with

$$N : = (I - L_B L_C)^{-1}, \tag{5}$$

$$W : = N L_B, \tag{6}$$

$$X : = L_C N, \tag{7}$$

we have $W = W^* \geq 0$, $X = X^* \geq 0$. Some nontrivial algebraic manipulations show that W and X satisfy the algebraic Riccati equations given below. We first define the following operators

$$T_X := (I + B^*XB)^{1/2}, \quad T_W := (I + CWC^*)^{1/2}. \tag{8}$$

Lemma 3.3. *The nonnegative selfadjoint operator X defined by* (7) *is a solution of the control algebraic Riccati equation of the system*

$$\left[\begin{array}{c|c} A & B \\ \hline T_W^{-1}CN & 0 \end{array}\right], \tag{9}$$

$$A^*XA - X + N^*CT_W^{-2}CN = A^*XBT_X^{-2}B^*XA.$$

The nonnegative selfadjoint operator W defined by (6) *is a solution of the filter algebraic Riccati equation of the system*

$$\left[\begin{array}{c|c} A & NBT_X^{-1} \\ \hline C & 0 \end{array}\right], \tag{10}$$

$$AWA^* - W + NBT_X^{-2}B^*N^* = AWC^*T_W^{-2}CWA^*. \tag{11}$$

Proof. We only give a proof the case of the system (10) since that for the system (9) is very similar. We first note that the following identities hold:

$$I + BB^*X = (I - AL_BA^*L_C)\,(I - L_BL_C)^{-1}, \tag{12}$$
$$I + C^*CW = (I - A^*L_CAL_B)\,(I - L_CL_B)^{-1}. \tag{13}$$

We will prove only (13) since the proof for (12) is similar. Using the definition (6) of W and the observation Lyapunov equation (3) we have

$$I + C^*CW = I + (L_C - A^*L_CA)\,L_B(I - L_CL_B)^{-1}.$$

This can be rewritten as

$$(I - L_CL_B + L_CL_B - A^*L_CAL_B)\,(I - L_CL_B)^{-1},$$

which simplifies to the right-hand side of (13).

Our goal is to show that (11) holds. We first note that by the definition (5) of N we have

$$NB(I + B^*XB)^{-1}B^*N^* = (I - L_BL_C)^{-1}(I + BB^*X)^{-1}BB^*(I - L_CL_B)^{-1}.$$

Next we use the control Lyapunov equation to rewrite this as

$$(I - L_BL_C)^{-1}(I + BB^*X)^{-1}\,(L_B - AL_BA^*)\,(I - L_CL_B)^{-1},$$

which by (12) in turn equals

$$(I - AL_BA^*L_C)^{-1}\,(L_B - AL_BA^*)\,(I - L_CL_B)^{-1}.$$

Rewriting this first as

$$(I - AL_BA^*L_C)^{-1}\,([I - AL_BA^*L_C]L_B - AL_BA^*[I - L_CL_B])\,(I - L_CL_B)^{-1}$$

and then simplifying this via

$$L_B(I - L_C L_B)^{-1} - (I - AL_B A^* L_C)^{-1}\, AL_B A^*$$

to

$$L_B(I - L_C L_B)^{-1} - AL_B\,(I - A^* L_C A L_B)^{-1}\, A^*$$

and using the definition (6) of W gives

$$W - AW(I - L_C L_B)(I - A^* L_C A L_B)^{-1} A^*.$$

We now use (13) and obtain for the above

$$W - AW(I + C^* C W)^{-1} A^*,$$

which we first rewrite as

$$W - AW(I + C^* C W)^{-1}\,(-C^* C W + I + C^* C W)\, A^*$$

and subsequently simplify to

$$W + AW(I + C^* C W)^{-1} C^* C W A^* - AWA^*.$$

So we finally obtain

$$NB(I + B^* X B)^{-1} B^* N^* = W + AW(I + C^* C W)^{-1} C^* C W A^* - AWA^*,$$

which is easily seen to be equivalent to the desired (11). □

From Lemmas 3.2 and 3.3 we obtain the following.

Lemma 3.4. *The system*

$$\left[\begin{array}{c|c} A_X & B \\ \hline -T_X^{-2} B^* X A & 0 \\ T_W^{-1} C N & 0 \end{array}\right], \tag{14}$$

where

$$A_X := (I + BB^* X)^{-1} A, \tag{15}$$

is output stable and input-output stable.

The system

$$\left[\begin{array}{c|cc} A_W & -AWC^* T_W^{-2} & NBT_X^{-1} \\ \hline C & 0 & 0 \end{array}\right], \tag{16}$$

where

$$A_W := A(I + WC^* C)^{-1}, \tag{17}$$

is input stable and input-output stable.

Proof. The system (14) is the closed-loop system (up to an insignificant similarity transformation in the input space) of the control Riccati equation for the system (9). So by Lemma 3.2 it is output stable and input-output stable. The argument for the system (16) is similar, but based on the filter Riccati equation for (10). $\square$

The following important result shows that N intertwines A_X and A_W.

Lemma 3.5. *The following identity holds:*

$$NA_X = A_W N. \tag{18}$$

Proof. Using the definitions of A_X, A_W and N we see that (18) is equivalent to

$$A(I - L_B L_C)(I + WC^*C) = (I + BB^*X)(I - L_B L_C)A,$$

and using $X = L_C N$ and $W = NL_B$ this in turn is equivalent to

$$A(I - L_B L_C + L_B C^*C) = (I + BB^*L_C - L_B L_C)A,$$

which is easily seen to be true by using $C^*C = L_C - A^*L_C A$ and $BB^* = L_B - AL_B A^*$. $\square$

The following formula that relates NA to AN will also be useful.

Lemma 3.6. *The following identity holds:*

$$NA = AN - NAL_B C^*CN + NBB^*L_C AN. \tag{19}$$

Proof. This follows easily after substituting $C^*C = L_C - A^*L_C A$ and $BB^* = L_B - AL_B A^*$. $\square$

4 J-spectral factorization

The following J-spectral factorization plays a crucial role.

$$\begin{bmatrix} I & \mathbf{G}^* \\ 0 & I \end{bmatrix} \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ \mathbf{G} & I \end{bmatrix} - \mathbf{X} \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \mathbf{X}^* \leq 0.$$

The objective is to find a analytic function on the open unit disc $\mathbf{X}$ such that the above inequality holds on the open unit disc. This function $\mathbf{X}$ is subsequently used to define a function $\mathbf{K}$ and the above inequality is used to show that this $\mathbf{K}$ is a solution of the suboptimal Nehari problem. It was first realized in [6] that it is this inequality that is important and not equality on the unit circle, which was used in all previous approaches via J-spectral factorization to the Nehari problem.

Our candidate solution is the following.

Definition 4.1. *Denote by* $\mathbf{X}$ *the transfer function and by* $\mathfrak{X}$ *the characteristic function of the system*

$$\left[\begin{array}{c|cc} A & -NBT_X^{-1} & AWC^*T_W^{-1} \\ \hline B^*L_CA & T_X^{-1} & B^*L_CAWC^*T_W^{-1} \\ C & 0 & T_W \end{array}\right]. \tag{20}$$

By an application of Lemma 2.3 we obtain the following stability result.

Lemma 4.2. *The system given by* (20) *is output stable.*

Proof. By our standing hypothesis $\Sigma(A,-,C,-)$ is input-output and output stable. It then immediately follows from Lemma 2.3 that $\Sigma(A,-,[B^*L_CA;C],-)$ is output stable and then the same conclusion can be made about the system (20). □

It will prove useful to have the following alternative formula for $\mathbf{X}$.

Lemma 4.3. *We have* $\mathbf{X} = \mathbf{X}_1 T$ *with* $\mathbf{X}_1$ *the transfer function of the system*

$$\left[\begin{array}{c|cc} A & -NB & NAL_BC^* \\ \hline B^*L_CA & I & 0 \\ C & 0 & I \end{array}\right] \tag{21}$$

and

$$T := \begin{bmatrix} T_X^{-1} & B^*L_CAWC^*T_W^{-1} \\ 0 & T_W \end{bmatrix}. \tag{22}$$

The corresponding result for characteristic functions also holds.

Proof. The only thing to prove is that

$$B_h := [-NB, NAL_BC^*]\,T = \left[-NBT_X^{-1}, AWC^*T_W^{-1}\right], \tag{23}$$

for the first component this is trivial and for the second component this amounts to proving that

$$-NBB^*L_CAWC^*T_W^{-1} + NAL_BC^*T_W = AWC^*T_W^{-1}. \tag{24}$$

Multiplying (19) from the right with L_BC^* and noting that $W = NL_B$ gives

$$NAL_BC^* = AWC^* - NAL_BC^*CWC^* + NBB^*L_CAWC^*,$$

which after rearranging and multiplication to the right with T_W^{-1} gives (24). □

We have the following useful identities involving the operator T.

Lemma 4.4. *With the notation*

$$J := \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

the following identities hold:

$$TJT^* = \begin{bmatrix} -I + B^*L_CB + B^*L_CAWA^*L_CB & B^*L_CAWC^* \\ CWA^*L_CB & T_W^2 \end{bmatrix}, \tag{25}$$

$$TJT^* \begin{bmatrix} -B^*N^* \\ CL_BA^*N^* \end{bmatrix} = \begin{bmatrix} B^* + B^*L_CAWA^* \\ CWA^* \end{bmatrix} \tag{26}$$

and

$$[-NB, NAL_BC^*]\, TJT^* \begin{bmatrix} -B^*N^* \\ CL_BA^*N^* \end{bmatrix} = AWA^* - W. \tag{27}$$

Proof. To prove (25) we only need to show that

$$-T_X^{-2} + B^*L_CAWC^*T_W^{-2}CWA^*L_CB = -I + B^*L_CB + B^*L_CAWA^*L_CB. \tag{28}$$

By the algebraic Riccati equation for W we have for the left-hand side of (28)

$$-T_X^{-2} + B^*L_C\left(AWA^* - W + NBT_X^{-2}B^*N^*\right)L_CB.$$

Thus (28) is equivalent to

$$-T_X^{-2} - B^*L_CWL_CB + B^*L_CNBT_X^{-2}B^*N^*L_CB = -I + B^*L_CB,$$

i.e. that

$$-T_X^2B^*L_CWL_CB + T_X^2B^*L_CNBT_X^{-2}B^*N^*L_CB + T_X^2 - T_X^2B^*L_CB = I.$$

Using that $L_CN = X$ and $L_CW = XL_B$ this is equivalent to

$$-T_X^2B^*XL_BL_CB + T_X^2B^*XBT_X^{-2}B^*XB + T_X^2 - T_X^2B^*L_CB = I.$$

Furthermore, we have $B^*XBT_X^{-2} = T_X^{-2}B^*XB$ so that the above is equivalent to

$$-T_X^2B^*XL_BL_CB + B^*XBB^*XB + T_X^2 - T_X^2B^*L_CB = I.$$

Using the definition of T_X from (8) this is equivalent to

$$\begin{aligned} &- B^*XL_BL_CB - B^*XBB^*XL_BL_CB + B^*XBB^*XB \\ &\quad + B^*XB - B^*L_CB - B^*XBB^*L_CB = 0. \end{aligned} \tag{29}$$

Combining the first and fourth term on the left-hand side gives

$$-B^*XL_BL_CB + B^*XB = B^*X(I - L_BL_C)B = B^*L_CB,$$

so that these terms cancel against the fifth term on the left-hand side of (29). The second and third term on the left-hand side of (29) are

$$\begin{aligned} &- B^*XBB^*XL_BL_CB + B^*XBB^*XB \\ &= B^*XBB^*X\left(I - L_BL_C\right)B = B^*XBB^*L_CB, \end{aligned}$$

so that they cancel against the sixth term on the left-hand side of (29). So (29) holds and consequently (28) and (25) hold.

Using (25) we see that the left-hand side of (26) equals

$$\begin{bmatrix} B^* - B^* L_C BB^* - B^* L_C AWA^* L_C BB^* + B^* L_C AWC^* CL_B A^* \\ -CWA^* L_C BB^* + CL_B A^* + CWC^* CL_B A^* \end{bmatrix} N^*. \quad (30)$$

Now using the Lyapunov equations we see that the first component of (30) equals

$$\begin{aligned} &B^* \left(I - L_C L_B + L_C AL_B A^* - L_C AWA^* L_C L_B + L_C AWA^* L_C AL_B A^* \right.\\ &\quad \left. + L_C AWL_C L_B A^* - L_C AWA^* L_C AL_B A^*\right) N^*. \end{aligned}$$

We see that the fifth and seventh terms cancel against each other. Using that $W = L_B N^*$ and the definition of N we obtain

$$B^* \left(I - L_C L_B + L_C AW \left[(I - L_C L_B)A^* - A^* L_C L_B + L_C L_B A^*)\right]\right) N^*.$$

Cancelling terms we obtain

$$B^* \left(I - L_C L_B + L_C AWA^* \left[I - L_C L_B\right]\right) N^*,$$

which equals

$$B^* + B^* L_C AWA^*,$$

as desired. Using that $W = L_B N^*$ and the definition of N we see that the second component of (30) equals

$$CW \left(-A^* L_C BB^* + (I - L_C L_B)A^* + C^* CL_B A^*\right) N^*.$$

Now using the Lyapunov equations we see that this equals

$$CWHN^*,$$

with

$$H := -A^* L_C L_B + A^* L_C AL_B A^* + (I - L_C L_B)A^* + L_C L_B A^* - A^* L_C AL_B A^*.$$

After cancellations we see that this equals

$$CW \left(-A^* L_C L_B + A^*\right) N^* = CWA^*,$$

as desired.

The equality (27) follows by recognizing that the left-hand side equals $B_h J B_h^*$ with

$$B_h = [-NBT_X^{-1}, AWC^* T_W^{-1}]$$

(compare (23)), so that (27) is nothing else than the Riccati equation (11). $\square$

The following lemma is proven by more algebraic manipulations.

Lemma 4.5. *On $1/\rho(A)$ we have*

$$\begin{bmatrix} I & \mathfrak{G}^* \\ 0 & I \end{bmatrix} \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ \mathfrak{G} & I \end{bmatrix} - \mathfrak{X} \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \mathfrak{X}^* \tag{31}$$
$$= \frac{|z|^2 - 1}{|z|^2} \left(\begin{bmatrix} \mathfrak{L} \\ \mathfrak{M} \end{bmatrix} \begin{bmatrix} \mathfrak{L}^* & \mathfrak{M}^* \end{bmatrix} + \begin{bmatrix} \mathfrak{L}_0^* \\ 0 \end{bmatrix} \begin{bmatrix} \mathfrak{L}_0 & 0 \end{bmatrix} \right),$$

where $\mathfrak{L}$ is the characteristic function of the system

$$\left[\begin{array}{c|c} A & W^{1/2} \\ \hline B^* L_C A & 0 \end{array}\right], \tag{32}$$

$\mathfrak{M}$ is the characteristic function of the system

$$\left[\begin{array}{c|c} A & W^{1/2} \\ \hline C & 0 \end{array}\right], \tag{33}$$

and $\mathfrak{L}_0$ is the characteristic function of the system

$$\left[\begin{array}{c|c} A & B \\ \hline L_C^{1/2} & 0 \end{array}\right]. \tag{34}$$

Proof. We have

$$\begin{aligned} \mathfrak{X} J \mathfrak{X}^* &= \mathfrak{X}_1 T J T^* \mathfrak{X}_1 \\ &= T J T^* + \begin{bmatrix} B^* L_C A \\ C \end{bmatrix} \left(\frac{1}{z} - A \right)^{-1} [-NB, NAL_B C^*] \, T^* J T \\ &\quad + T J T^* \begin{bmatrix} -B^* N^* \\ C L_B A^* N^* \end{bmatrix} \left(\frac{1}{z} - A \right)^{-*} [A^* L_C B, C^*] \\ &\quad + \begin{bmatrix} B^* L_C A \\ C \end{bmatrix} \left(\frac{1}{z} - A \right)^{-1} [-NB, NAL_B C^*] \, T^* J T \\ &\quad \times \begin{bmatrix} -B^* N^* \\ C L_B A^* N^* \end{bmatrix} \left(\frac{1}{z} - A \right)^{-*} [A^* L_C B, C^*] \, . \end{aligned}$$

Using (25), (26) and (27) we see that the right-hand side equals

$$\begin{aligned} &\begin{bmatrix} -I + B^* L_C B + B^* L_C A W A^* L_C B & B^* L_C A W C^* \\ C W A^* L_C B & T_W^2 \end{bmatrix} \\ &\quad + \begin{bmatrix} B^* L_C A \\ C \end{bmatrix} \left(\frac{1}{z} - A \right)^{-1} [B + A W A^* L_C B, A W C^*] \\ &\quad + \begin{bmatrix} B^* + B^* L_C A W A^* \\ C W A^* \end{bmatrix} \left(\frac{1}{z} - A \right)^{-*} [A^* L_C B, C^*] \\ &\quad + \begin{bmatrix} B^* L_C A \\ C \end{bmatrix} \left(\frac{1}{z} - A \right)^{-1} (A W A^* - W) \left(\frac{1}{z} - A \right)^{-*} [A^* L_C B, C^*] \, . \end{aligned} \tag{35}$$

We proceed by considering the separate components of this 2 by 2 matrix.

The (1,1) component of (35) equals

$$\begin{aligned}
&-I + B^* L_C B + B^* L_C A \left(\frac{1}{z} - A\right)^{-1} B + B^* \left(\frac{1}{z} - A\right)^{-*} A L_C B \\
&+ \left(\frac{1}{|z|^2} - 1\right) B^* L_C A \left(\frac{1}{z} - A\right)^{-1} W \left(\frac{1}{z} - A\right)^{-*} A^* L_C B,
\end{aligned}$$

where we have used the identity

$$\begin{aligned}
&\left(\frac{1}{|z|^2} - 1\right) W \\
&= \left(\frac{1}{z} - A\right) W \left(\frac{1}{z} - A\right)^* + AW \left(\frac{1}{z} - A\right)^* + \left(\frac{1}{z} - A\right) W A^* + AWA^* - W.
\end{aligned}$$

Using a similar calculation we obtain

$$\begin{aligned}
\mathfrak{G}^* \mathfrak{G} &= B^* \left(\frac{1}{z} - A\right)^{-*} C^* C \left(\frac{1}{z} - A\right)^{-1} B \\
&= B^* \left(\frac{1}{z} - A\right)^{-*} \\
&\quad \times \left[\left(\frac{1}{z} - A\right)^* L_C A + \left(\frac{1}{z} - A\right)^* L_C \left(\frac{1}{z} - A\right)\right. \\
&\qquad\qquad \left. + A^* L_C \left(\frac{1}{z} - A\right) + \left(1 - \frac{1}{|z|^2}\right) L_C\right] \\
&\quad \times \left(\frac{1}{z} - A\right)^{-1} B \\
&= B^* L_C A \left(\frac{1}{z} - A\right)^{-1} B + B^* L_C B + \left(\frac{1}{z} - A\right)^{-*} A^* L_C B \\
&\quad + \left(1 - \frac{1}{|z|^2}\right) B^* \left(\frac{1}{z} - A\right)^{-*} L_C \left(\frac{1}{z} - A\right)^{-1} B.
\end{aligned}$$

where we have used the observation Lyapunov equation.

It follows from the formulas obtained for $\mathfrak{X} J \mathfrak{X}^*$ and $\mathfrak{G}^* \mathfrak{G}$ that the (1,1) component of the left-hand side of (31) equals

$$\left(1 - \frac{1}{|z|^2}\right) (H_1 + H_2),$$

with

$$\begin{aligned}
H_1 &= B^* \left(\frac{1}{z} - A\right)^{-*} L_C \left(\frac{1}{z} - A\right)^{-1} B, \\
H_2 &= B^* L_C A \left(\frac{1}{z} - A\right)^{-1} W \left(\frac{1}{z} - A\right)^{-*} A^* L_C B,
\end{aligned}$$

as desired. The proof for the other components is obtained by similar tedious calculations and hence is omitted. $\square$

The following lemma gives the corresponding result for transfer functions.

Lemma 4.6. *On the open unit disc we have*

$$\begin{bmatrix} I & \mathbf{G}^* \\ 0 & I \end{bmatrix} \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ \mathbf{G} & I \end{bmatrix} - \mathbf{X} \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \mathbf{X}^*$$
$$= \frac{|z|^2 - 1}{|z|^2} \left(\begin{bmatrix} \mathbf{L} \\ \mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{L}^* & \mathbf{M}^* \end{bmatrix} + \begin{bmatrix} \mathbf{L}_0^* \\ 0 \end{bmatrix} \begin{bmatrix} \mathbf{L}_0 & 0 \end{bmatrix} \right),$$

where $\mathbf{L}$ *is the transfer function of the system* (32)*,* $\mathbf{M}$ *is the transfer function of the system* (33) *and* $\mathbf{L}_0$ *is the transfer function of the system* (34)*.*

Proof. From Lemma 4.5 we obtain equality of the transfer functions in a neighborhood of zero. The transfer functions involved are all analytic on the open unit disc: for $\mathbf{G}$, $\mathbf{M}$ and $\mathbf{L}_0$ this follows directly from our stability assumptions, while for $\mathbf{X}$ this follows from Lemma 4.2 and for $\mathbf{L}$ this follows using Lemma 2.3. The left- and right-hand sides of the equation are not analytic on the unit disc, but they are real-analytic there. This last assertion follows from the fact that both analytic functions and their adjoints are real-analytic and that the product of real-analytic functions is again real-analytic. By the identity theorem for real-analytic functions the equality holds on the whole unit disc (see [5, Appendix]). $\square$

Remark 4.7. *We note that the right-hand sides of the equations obtained in Lemmas 4.5 and 4.6 are strictly speaking not defined for* $z = 0$*, since we divide by* $|z|^2$*. Noting that the feedthrough terms of the systems* (32)*,* (33) *and* (34) *are zero this singularity is seen to be removable, i.e. the functions can be continuously (and even analytically) extended to* $z = 0$*. It is in this sense that the equalities hold in* $z = 0$*.*

From Lemma 4.6 we deduce the following.

Lemma 4.8. *On the open unit disc we have*

$$\begin{bmatrix} I & \mathbf{G}^* \\ 0 & I \end{bmatrix} \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ \mathbf{G} & I \end{bmatrix} \leq \mathbf{X} \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \mathbf{X}^*.$$

Proof. The term between brackets on the right-hand side of the equation obtained in Lemma 4.6 is of the form $T_1^* T_1 + T_2^* T_2$ and so it is nonnegative. The fraction in front, $\frac{|z|^2-1}{|z|^2}$, is negative since by assumption $|z| < 1$. It follows that the right-hand side of the equation obtained in Lemma 4.6 is, for every $z \in \mathbb{D}$, nonpositive. It follows that the left-hand side is, which gives the desired inequality. $\square$

The following lemma gives important formulas for the inverse of one of the components of $\mathbf{X}$.

Lemma 4.9. *The inverse of the* $(2,2)$ *component of* $\mathbf{X}$ *on the open unit disc is the transfer function of the input stable and input-output stable system*

$$\left[\begin{array}{c|c} A_W & AWC^*T_W^{-2} \\ \hline -T_W^{-1}C & T_W^{-1} \end{array}\right]. \tag{36}$$

Proof. That the characteristic functions of (36) is the inverse of that of the (2,2) component of (20) follows by an application of Lemma 2.1 together with the identity

$$A - AWC^*T_W^{-2}C = A_W,$$

which is easily proven using only the definition of T_W.

That the system (36) is input and input-output stable follows from Lemma 3.4.

Since the characteristic functions of the two systems are inverses it follows that the product of the two transfer functions equals the identity for z in a neighborhood of zero. This equality extends to the whole of the open unit disc since these transfer functions are analytic on the open unit disc by the stability properties of the given realizations. □

5 The inverse of the spectral factor

Definition 5.1. *Define* $\mathbf{V}$ *as the transfer function and* $\mathfrak{V}$ *as the characteristic function of the system*

$$\left[\begin{array}{c|cc} A & B & -AL_BC^* \\ \hline T_X^{-1}B^*XA & T_X & -T_X^{-1}B^*XAL_BC^* \\ T_W^{-1}CN & 0 & T_W^{-1} \end{array}\right]. \tag{37}$$

Lemma 5.2. *The system given by* (37) *is input stable.*

Proof. This follows similarly as Lemma 4.2. □

Lemma 5.3. *We have* $\mathbf{V} = T^{-1}\mathbf{V}_1$ *with* $\mathbf{V}_1$ *the transfer function of the system*

$$\left[\begin{array}{c|cc} A & B & -AL_BC^* \\ \hline B^*L_CAN & I & 0 \\ CN & 0 & I \end{array}\right] \tag{38}$$

and

$$T^{-1} = \begin{bmatrix} T_X & -T_X^{-1}B^*XAL_BC^* \\ 0 & T_W^{-1} \end{bmatrix} \tag{39}$$

the inverse of the operator T *defined by* (22). *The corresponding result for characteristic functions also holds.*

Proof. We first show that the right-hand side of (39) is indeed T^{-1}. It is easily seen that the inverse of T defined by (22) is given by

$$\begin{bmatrix} T_X & -T_X B^* L_C A W C^* T_W^{-2} \\ 0 & T_W^{-1} \end{bmatrix}. \tag{40}$$

To show that this equals (39) we need to show that

$$T_X B^* L_C A W C^* T_W^{-2} = T_X^{-1} B^* X A L_B C^*,$$

or equivalently that

$$B^* L_C A W C^* = T_X^{-2} B^* X A L_B C^* T_W^2.$$

Since $W = N L_B$, with (19) we see that the left-hand side equals

$$B^* L_C N \left[A L_B + A L_B C^* C N L_B - B B^* L_C A N L_B \right] C^*,$$

which by using $X = L_C N$ and $W = N L_B$ is seen to equal

$$B^* X \left[A N^{-1} - B B^* X N^{-1} A (I + W C^* C)^{-1} \right] (I + W C^* C) W C^*.$$

Using (15) and (17) this can be rewritten as

$$B^* X (I + B B^* X) \left[A_X N^{-1} - (I + B B^* X)^{-1} B B^* X N^{-1} A_W \right] W T_W^2 C^*.$$

From (18) we have $A_X N^{-1} = N^{-1} A_W$ so that the above equals

$$B^* X (I + B B^* X) \left[I - (I + B B^* X)^{-1} B B^* X \right] A_X N^{-1} W T_W^2 C^*.$$

Simplifying shows that this equals

$$B^* X A_X L_B T_W^2 C^*,$$

and using (15) shows that this in turn equals

$$T_X^{-2} B^* X A L_B T_W^2 C^*,$$

as desired.

To show that $\mathbf{V} = T \mathbf{V}_1$ it only remains to show that

$$T_X^{-1} B^* X A = T_X B^* L_C A N - T_X^{-1} B^* X A L_B C^* C N.$$

After multiplying (19) from the left by $B^* L_C$ we obtain

$$B^* L_C N A = B^* L_C A N - B^* L_C N A L_B C^* C N + B^* L_C N B B^* L_C A N.$$

But $L_C N = X$ and so rearranging we obtain

$$B^* X A = (I + B^* X B) B^* L_C A N - B^* X A L_B C^* C N.$$

Multplication from the left by T_X^{-1} gives the desired equality. $\square$

The following is the analogue of Lemma 4.9 and is proven similarly.

Lemma 5.4. *The inverse of the* $(1,1)$ *component of* $\mathbf{V}$ *on the open unit disc is the transfer function of the output stable and input-output stable system*

$$\left[\begin{array}{c|c} A_X & -BT_X^{-1} \\ \hline T_X^{-2}B^*XA & T_X^{-1} \end{array}\right]. \tag{41}$$

Proof. That the characteristic function of the system (41) is the inverse of that of the $(1,1)$ component of (37) follows by an application of Lemma 2.1 together with the identity

$$A - BT_X^{-2}B^*XA = A_X,$$

which is easily proven using only the definition of T_X.

That the system (41) is output and input-output stable follows from Lemma 3.4.

Since the characteristic functions of the two systems are inverses it follows that the product of the two transfer functions equals the identity for z in a neighborhood of zero. This equality extends to the whole of the open unit disc since these transfer functions are analytic on the open unit disc by the stability properties of the given realizations. $\square$

The following lemma can be proven by algebraic manipulation.

Lemma 5.5. *On* $1/\rho(A)$ *we have* $\mathfrak{V}\mathfrak{X} = I = \mathfrak{X}\mathfrak{V}$, *where* $\mathfrak{X}$ *is the characteristic function of the system* (21) *and* $\mathfrak{V}$ *is the characteristic function of the system* (38).

Proof. We first note that equivalently we may show that the characteristic functions of the systems (21) and (38) are each others inverses.

Define

$$B_l := [-NB, NAL_BC^*], \quad C_l := \begin{bmatrix} B^*L_CA \\ C \end{bmatrix}.$$

We apply Lemma 2.1 to obtain the inverse of the characteristic function of the system (21). This is the characteristic function of the system

$$\left[\begin{array}{c|c} A - B_lC_l & B_l \\ \hline -C_l & I \end{array}\right]. \tag{42}$$

The system (38) may be written in terms of B_l and C_l as

$$\left[\begin{array}{c|c} A & -N^{-1}B_l \\ \hline C_lN & I \end{array}\right], \tag{43}$$

provided that

$$A - B_lC_l = NAN^{-1}.$$

But after multiplication from the right by N and substituting the definitions of B_l and C_l this simplifies to (19) $\square$

The case of transfer functions follows immediately.

Lemma 5.6. *On the open unit disc we have* $\mathbf{V}\mathbf{X} = I = \mathbf{X}\mathbf{V}$*, where* $\mathbf{X}$ *is the transfer function of the system* (21) *and* $\mathbf{V}$ *is the transfer function of the system* (38)*.*

Proof. Using Lemma 5.5 for the corresponding result for characteristic functions, the output stability of the system (21) and the input stability of the system (38) (which follows from the dual version of Lemma 2.3) we obtain the desired result. □

From the previous lemma and the J-spectral factorization inequality obtain in Lemma 4.8 we obtain the following.

Lemma 5.7. *On the open unit disc we have*

$$\mathbf{V}\begin{bmatrix} I & \mathbf{G}^* \\ 0 & I \end{bmatrix}\begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} I & 0 \\ \mathbf{G} & I \end{bmatrix}\mathbf{V}^* \leq \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}.$$

Proof. This follows from multiplying both sides of the inequality obtained in Lemma 4.8 from the left with $\mathbf{V}$, from the right with $\mathbf{V}^*$ and using that $\mathbf{V}$ is the inverse of $\mathbf{X}$ by Lemma 5.6. □

6 The Nehari problem

In this section we give an explicit solution to our Nehari problem using the inverse J-spectral factor $\mathbf{V}$ found in Section 5.

Definition 6.1. *Define* $\mathbf{Z}$ *as the transfer function of the system*

$$\left[\begin{array}{c|c} A_X^* & -A_X^* X B \\ \hline C L_B A_X^* & -C L_B A_X^* X B \end{array}\right]. \tag{44}$$

Lemma 6.2. *The system* (44) *is input stable.*

Proof. Since $A_X = (I + BB^*X)^{-1}A$, we see that the adjoint of the control operator, B^*XA_X, equals $(I + B^*XB)^{-1}B^*XA$. Using Lemma 3.4, specifically that the system (14) is output stable, we obtain the desired result. □

The following can be proven by algebraic manipulation.

Lemma 6.3. *The characteristic function of the system* (44) *equals the characteristic function of the system*

$$\left[\begin{array}{c|c} A_W^* & -A_W^* L_C B \\ \hline C W A_W^* & -C W A_W^* L_C B \end{array}\right]. \tag{45}$$

Proof. From (18) it follows by multiplication from the left with L_C and from the right with L_B that $XA_XL_B = L_CA_WW$, which in turn implies that the right-bottom terms in (44) and (45) are equal. Using (18) it follows that the top right-hand corner of (44) equals that of (45) multiplied from the left by N^*, the bottom left-hand corner of (44) equals that of (45) multiplied from the right by N^{-*} and the top left-hand corner of (44) equals that of (45) multiplied from the right by N^{-*} and from the left by N^*. In the calculation of the characteristic functions this causes cancellation, showing that the characteristic functions are equal. □

The following can be proven in a similar manner as Lemma 6.2.

Lemma 6.4. *The system* (45) *is output stable.*

Lemma 6.5. *The transfer functions of the systems* (44) *and* (45) *are equal on the open unit disc.*

Proof. Since the system (44) is input stable, its transfer function is analytic on the open unit disc. Since the system (45) is output stable, the same holds for this system. By Lemma 6.3 the characteristic functions of these systems are equal. So the transfer functions restricted to the open unit disc are analytic extensions of the same function. By the identity theorem for analytic functions they must be equal. □

The following lemmas relate $\mathbf{Z}$ to the J-spectral factor $\mathbf{X}$ and its inverse $\mathbf{V}$. The first of these lemmas deals with the corresponding characteristic functions and can by proven by algebraic manipulation.

Lemma 6.6. *On* $1/\rho(A^*) \cap 1/\rho(A_X^*)$ *we have*

$$\mathfrak{Z}(z) = \mathfrak{V}_{12}(\bar{z})^* \mathfrak{V}_{11}(\bar{z})^{-*}. \tag{46}$$

On $1/\rho(A^*) \cap 1/\rho(A_W^*)$ *we have*

$$\mathfrak{Z}(z) = -\mathfrak{X}_{22}(\bar{z})^{-*} \mathfrak{X}_{21}(\bar{z})^*. \tag{47}$$

Proof. We first note that $T_X^{-2}B^*XA = B^*XA_X$, and that if follows from this equality using Lemma 3.5 that $T_X^{-2}B^*XA = B^*L_CA_WN$.

Using Definition 5.1, Lemma 5.4 and the identity $T_X^{-2}B^*XA = B^*L_CA_WN$, we have

$$\begin{aligned}
&\mathfrak{V}_{11}(z)^{-1}\mathfrak{V}_{12}(z) \\
&= -B^*L_CA_WWC^* + T_X^{-2}B^*XA\left(\frac{1}{z}I - A_X\right)^{-1} BB^*L_CA_WWC^* \\
&\quad + T_X^{-2}B^*XA\left(\frac{1}{z}I - A_X\right)^{-1}\left[-\left(\frac{1}{z}I - A_X\right) + BT_X^{-2}B^*XA\right] \\
&\quad \times \left(\frac{1}{z}I - A\right)^{-1} AL_BC^*.
\end{aligned}$$

The term in square brackets is easily seen to equal $-\left(\frac{1}{z}I - A\right)$, so that the above simplifies to

$$-B^* L_C A_W W C^* + T_X^{-2} B^* X A \left(\frac{1}{z}I - A_X\right)^{-1} \left[B B^* L_C A_W W C^* - A L_B C^*\right].$$

The term in square brackets simplifies to $-A_X L_B C^*$. Using this, the identity $T_X^{-2} B^* X A = B^* X A_X$ and comparing to the formula for $\mathfrak{Z}$ in (44) (and (45) for the constant term) we obtain (46). The equality (47) is proven similarly. $\square$

Lemma 6.7. *On the open unit disc we have*

$$\mathbf{Z}(z) = \mathbf{V}_{12}(\bar{z})^* \mathbf{V}_{11}(\bar{z})^{-*} = -\mathbf{X}_{22}(\bar{z})^{-*} \mathbf{X}_{21}(\bar{z})^*. \tag{48}$$

Proof. This follows from Lemma 6.6 and the stability properties of the given realizations of the involved transfer functions. $\square$

The following is the main result of this article.

Theorem 6.8. *Define* $\mathbf{K}(z) := \mathbf{Z}(1/z)$, *where* $\mathbf{Z}$ *is defined as in Definition 6.1. Then* $\mathbf{K} \in \mathbf{H}_\infty(\mathbb{D}^+, \mathcal{L}(U,Y))$ *and* $\|\mathbf{G} + \mathbf{K}\|_{\mathbf{L}_\infty(\mathbb{T},\mathcal{L}(U,Y))} \leq 1$.

Proof. Noting that all the transfer functions $\mathbf{G}$, $\mathbf{V}$, and $\mathbf{Z}$ are holomorphic on $\mathbb{D}$, we perform some elementary calculations on $\mathbb{D}$. We first verify that

$$\begin{bmatrix} I \\ \mathbf{G}(z) + \mathbf{Z}(\bar{z}) \end{bmatrix} = \begin{bmatrix} I & 0 \\ \mathbf{G}(z) & I \end{bmatrix} \mathbf{V}(z)^* \begin{bmatrix} \mathbf{V}_{11}(z)^{-*} \\ 0 \end{bmatrix}$$

by expanding the right hand-side:

$$\begin{aligned} &\begin{bmatrix} I & 0 \\ \mathbf{G}(z) & I \end{bmatrix} \begin{bmatrix} \mathbf{V}_{11}(z)^* & \mathbf{V}_{21}(z)^* \\ \mathbf{V}_{12}(z)^* & \mathbf{V}_{22}(z)^* \end{bmatrix} \begin{bmatrix} \mathbf{V}_{11}(z)^{-*} \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ \mathbf{G}(z) & I \end{bmatrix} \begin{bmatrix} I \\ \mathbf{V}_{12}(z)^* \mathbf{V}_{11}(z)^{-*} \end{bmatrix} \\ &= \begin{bmatrix} I \\ \mathbf{G}(z) + \mathbf{Z}(\bar{z}) \end{bmatrix}, \end{aligned}$$

where by Lemma 6.7

$$\mathbf{Z}(\bar{z}) = \mathbf{V}_{12}(z)^* \mathbf{V}_{11}(z)^{-*}$$

holds on the open unit disc. Using the just established identity we have

$$\begin{aligned} &(\mathbf{G}(z) + \mathbf{Z}(\bar{z}))^* (\mathbf{G}(z) + \mathbf{Z}(\bar{z})) - I \\ &= \begin{bmatrix} I \\ \mathbf{G}(z) + \mathbf{Z}(\bar{z}) \end{bmatrix}^* \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I \\ \mathbf{G}(z) + \mathbf{Z}(\bar{z}) \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{V}_{11}(z)^{-*} \\ 0 \end{bmatrix}^* \mathbf{V}(z) \begin{bmatrix} I & 0 \\ \mathbf{G}(z) & I \end{bmatrix}^* \times \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \\ &\quad \times \begin{bmatrix} I & 0 \\ \mathbf{G}(z) & I \end{bmatrix} \mathbf{V}(z)^* \begin{bmatrix} \mathbf{V}_{11}(z)^{-*} \\ 0 \end{bmatrix}. \end{aligned} \tag{49}$$

Applying Lemma 5.7 we obtain

$$\begin{aligned}
&(\mathbf{G}(z)+\mathbf{Z}(\bar{z}))^*\,(\mathbf{G}(z)+\mathbf{Z}(\bar{z}))-I \\
&\leq \begin{bmatrix} \mathbf{V}_{11}(z)^{-*} \\ 0 \end{bmatrix}^* \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \mathbf{V}_{11}(z)^{-*} \\ 0 \end{bmatrix} \\
&= -\mathbf{V}_{11}(z)^{-1}\mathbf{V}_{11}(z)^{-*} \leq 0.
\end{aligned} \tag{50}$$

This shows that the analytic function $\mathbf{Z}$ is bounded in norm on $\mathbb{D}$ and so it is in $\mathbf{H}_\infty(\mathbb{D}, \mathcal{L}(U,Y))$. We also obtain the estimate $\|\mathbf{G}(z)+\mathbf{Z}(\bar{z})\| \leq 1$ for $z \in \mathbb{D}$. By taking nontangential limits we obtain the same estimate almost everywhere on the unit circle. It follows from $\mathbf{Z} \in \mathbf{H}_\infty(\mathbb{D}, \mathcal{L}(U,Y))$ that $\mathbf{K}(z) = \mathbf{Z}(1/z)$ is in $\mathbf{H}_\infty(\mathbb{D}^+; \mathcal{L}(U,Y))$. The boundary functions of $\mathbf{Z}$ and $\mathbf{K}$ are related by $\mathbf{K}(\zeta) = \mathbf{Z}(\bar{\zeta})$ with ζ on the unit circle. So $\mathbf{K}$ satisfies the following for almost all ζ on the unit circle

$$\|\mathbf{G}(\zeta)+\mathbf{K}(\zeta)\| \leq 1. \qquad \square$$

Remark 6.9. *Theorem 6.8 gives one solution of the sub-optimal Nehari problem. With little extra effort one can actually obtain infinitely many. To show this we first note that* $\mathbf{V}_{21}\mathbf{V}_{11}^{-1}$ *equals the second component of the transfer function of the closed-loop system* (*as defined by* (4)) *of the system* (9)*. The proof is very similar to that of Lemma 6.6. It follows from Lemma 3.2 that* $\|\mathbf{V}_{21}\mathbf{V}_{11}^{-1}\|_\infty \leq 1$*. So if* $\|\mathbf{Q}\|_\infty < 1$*, then* $\mathbf{V}_{11}+\mathbf{Q}\mathbf{V}_{21}$ *has a well-defined inverse in* $\mathbf{H}_\infty$*. Now define* $\mathbf{Z}(\bar{z}) = (\mathbf{V}_{12}(z)^* + \mathbf{V}_{22}(z)^*\mathbf{Q}(z)^*)(\mathbf{V}_{11}(z)^* + \mathbf{V}_{21}(z)^*\mathbf{Q}(z)^*)^{-1}$*. Entirely analogously to the proof of Theorem 6.8 it follows that* $\mathbf{K}$ *defined by* $\mathbf{K}(z) = \mathbf{Z}(1/z)$ *is a solution of the suboptimal Nehari problem* (*Theorem 6.8 being the special case* $\mathbf{Q} = 0$)*. In the analogue of inequality* (50) *we used that* $\|\mathbf{Q}\|_\infty \leq 1$ (*strict inequality is not needed for this*)*.*

So for any $\mathbf{Q} \in \mathbf{H}_\infty(\mathbb{D}, \mathcal{L}(U,Y))$ *with* $\|\mathbf{Q}\|_\infty < 1$ *we obtain a solution.*

From the analogy with similar formulas known to give a parametrization of all solutions of the suboptimal Nehari problem, one would expect that the above given linear fractional formula (*possibly rewritten in Redheffer form*) *with parameter* $\mathbf{Q}$ *ranging over the closed unit ball* $\|\mathbf{Q}\|_\infty \leq 1$ *should give rise to all solutions. However, justification of such a statement would require further work beyond the scope of the present article.*

7 The general case

In this section we make some elementary observations that allow us to obtain the general case of the suboptimal Nehari problem from the special with $D = 0$ and $\sigma = 1$ considered above.

Remark 7.1. *If* $\mathbf{K}$ *is a solution of the suboptimal Nehari problem for data* $\mathbf{G} - \mathbf{G}(0)$*, then* $\mathbf{K} + \mathbf{G}(0)$ *is a solution of the suboptimal Nehari problem for data* $\mathbf{G}$*.*

Let $a, b > 0$. If $\mathbf{K}$ is a solution of the suboptimal Nehari problem for data $\mathbf{G}/a$ and parameter b, then $a\mathbf{K}$ is a solution of the suboptimal Nehari problem for data $\mathbf{G}$ and parameter ab.

If $S := \Sigma(A, B, C, D)$ is a realization of $\mathbf{G}$, then $S_a := \Sigma(A, B, aC, aD)$ is a realization of $a\mathbf{G}$. If L_C is the observability gramian of S, then $a^2 L_C$ is the observability gramian of S_a.

From the above remark and Theorem 6.8 it follows that if $\mathbf{G}$ is the transfer function of the state space system $\Sigma(A, B, C, D)$ and $\sigma > \|H_{\mathbf{G}}\|$, then the function $\mathbf{K}$ defined for $|z| > r(A_{X_\sigma})$ by

$$\mathbf{K}(z) = -D - CL_B A^*_{X_\sigma} X_\sigma B - CL_B A^*_{X_\sigma}(z - A^*_{X_\sigma})^{-1} A^*_{X_\sigma} X_\sigma B$$

extends to $\mathbf{K} \in \mathbf{H}_\infty(\mathbb{D}^+, \mathcal{L}(U, Y))$ which satisfies $\|\mathbf{G} + \mathbf{K}\|_{\mathbf{L}_\infty(\mathbb{T}, \mathcal{L}(U,Y))} \leq \sigma$. Here

$$X_\sigma = (\sigma^2 I - L_C L_B)^{-1} L_C,$$
$$A_{X_\sigma} = (I + BB^* X_\sigma)^{-1} A.$$

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